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A direct theory of viscous fluid flow in pipes I. Basic general developments

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This paper is concerned with the construction by a direct approach of a fairly general nonlinear theory of viscous fluid flow in both straight and curved pipes. First, a new procedure is used to establish a 1–1 correspondence between the lagrangian and eulerian formulations of the integral balance laws of a general thermomechanical theory of *directed* (or *Cosserat*) curve in the presence of a number of directors. (Such correspondence between lagrangian and eulerian formulations was previously possible in the special case of two directors with the resulting theory appropriate, for example, only for free surface jets.) The utility of the basic theory, which is valid for both compressible and incompressible linear viscous fluids, is then demonstrated with reference to the unsteady flow of an incompressible viscous fluid in straight pipes of varying elliptical cross section. A general solution is obtained for unsteady flow in straight pipes of elliptical cross section, and is applied to the cases in which a swirling motion is superposed on a uniform axial flow and a flow which is symmetric about the major and minor axes of the elliptical cross section.

1. Introduction

This paper is concerned with the construction by a direct approach of a fairly general nonlinear theory of linear (newtonian) viscous fluid flow applicable to both straight and curved pipes, which are not necessarily slender and which are of arbitrary cross section. The present paper (hereafter frequently referred to as Part I) deals mainly with the basic developments, while a companion paper (Part II under the same title) is devoted to application of the basic theory to viscous flow in curved pipes.

By way of background, we recall that the flow of viscous fluids in pipes, which includes Poiseuille and related flows, has long been a subject of intensive study. In fact, there seems to be a continued interest in flow of viscous fluids in straight pipes or tubes of variable cross section (see, for example, Duck 1978 & Hall 1974) and in curved pipes for which there is already an extensive theoretical and experimental literature cited by Berger *et al.* (1983).

The foregoing problems are analogous to those which arise in the study of slender rods in solid mechanics, where the procedure is to replace the three-dimensional system of equations with a two-dimensional system by approximations. In recent years, an approach based on a (three-dimensional) continuum *model* – called *directed* (or *Cosserat*) *curves* – has been developed in which an hierarchy of theories may be generated directly by postulates similar to those used in the construction of the three-dimensional theories. The continuum model comprises a space curve with any

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number of directors, and each of the hierarchy of theories gives rise to partial differential equations which, apart from dependence on time, depend only on a *single* independent coordinate variable along the space curve; for brevity and identification purposes, temporarily we refer to such directed curves as one-variable theories. (For additional background information, historical development of the theory of Cosserat (or directed) curves, and further references on the subject, see Naghdi (1982, p. 57), where a detailed definition for a rod-like body is given.)

We recall that a rod or jet is a special three-dimensional body whose boundary surface has special features. For the purpose of these background remarks, it may be defined as follows (Naghdi 1982). Consider a space curve – called a reference curve – embedded in a three-dimensional euclidean space. We refer to this curve in the current configuration by c. At each point of c imagine material filaments lying in the normal plane, i.e. the plane perpendicular to the tangent vector to c, and forming the normal cross section A_n. (The normal cross section of a jet (or a pipe) is a portion of the normal plane to the curve c, i.e. the intersection of the body and the normal plane.) The surface swept out by the closed boundary curve ∂A_n of A_n is called the lateral surface. Such a three-dimensional body, depending on the nature of application, may be referred to as rod-like or jet-like (or even tube-like) if the dimensions in the plane of the normal cross section are *small* compared to some characteristic dimension $L(c)$ of c, e.g. its local radius of curvature, or the length of c in the case of a straight curve. A tube-like (or jet-like) body is said to be *slender* if the largest dimension of A_n is much smaller than $L(c)$.

Although it is an interesting question, the justification for the relevance of this type of direct theory is not in regarding them as approximations to three-dimensional equations, but rather in their use as independent theories to predict some of the main properties of three-dimensional rod-like and jet-like (or tube-like) problems. Applications of a direct one-variable theory, or developments motivated by it, both in the context of inviscid and viscous fluid jets, have been carried out for a variety of problems by Green & Laws (1968), by Green (1975, 1976, 1977), by Caulk & Naghdi (1979*a*, *b*) and by Naghdi (1979) for Poiseuille flow in a uniform pipe of circular cross section and by Caulk & Naghdi (1987) for axisymmetric motion of a viscous fluid inside a slender surface of revolution with application to a number of problems involving both steady and unsteady flows.

The theory of directed (or Cosserat) curves was developed originally in the presence of two directors by Green & Laws (1966), Green *et al.* (1974), and extended subsequently in a lagrangian formulation and in the context of non-newtonian flow problems to the general case of K directors (Naghdi 1979). The main purpose of the present paper is to modify the previous developments in a manner which renders the results applicable to the flow of viscous fluid in pipes or tubes which are straight or curved and which may be of arbitrary cross sections. As the extension of the earlier results in the case of jets (Green 1976; Caulk & Naghdi 1979*a*) to corresponding problems for flow in pipes poses some difficulties, some modifications must be introduced in the existing formulation of the theory of direct curves.

In a different context, it was noted recently by Green & Naghdi (1984) that a common formulation (in the three-dimensional theory) uses material volumes and involves material time derivative from a lagrangian viewpoint, i.e. time differentiation holding the particle fixed. Subsequently, in the resulting local equations of motion, quantities involving material time derivatives can be expressed in eulerian form. Equivalently, the three-dimensional theory can be developed by using *fixed* volumes with corresponding local equations obtained directly in eulerian form. As is well known, the two approaches yield identical results in an exact three-dimensional theory. On the other hand, in previous developments of exact direct theories of fluid jets or rod-like bodies the basic equations have been stated using material surface areas, with local equations in lagrangian form. While the conversion of these local equations in the special cases considered previously to an eulerian form was fairly straightforward (Green & Laws 1968; Green 1976; Naghdi 1979), in general the conversion to this form poses some difficulty.

To overcome the difficulty referred to above, we need to develop a procedure which gives a basis for recasting the conservation laws in an alternative eulerian form. Starting with the three-dimensional equations this is carried out in Appendix A (placed following §3) and provides motivation for the desired statement of one-variable balance laws. With this background, in §2 we postulate one-variable conservation laws appropriate for Cosserat curve $\mathscr{R}_K$ – comprising a material curve with K ($K \geqslant 2$) directors – which leads directly to local equations in either eulerian or lagrangian form. The development in §2 is fairly general and includes thermal effects, although in the rest of this paper (Part I) and in Part II only the basic equations for an isothermal theory are utilized. (The inclusion of thermal effects occupies very little additional space and permits succinct simultaneous general presentations of both lagrangian and eulerian forms of the basic theory.) Finally, in §3, the basic theory is specialized for application to unsteady flow of an incompressible linear viscous fluid in a straight pipe of varying elliptical cross sections. A general analytical solution is obtained in the first part of §3 and a special case of this development is then applied (following (3.23)) to the problems of swirling on uniform axial flow and a flow (without swirling) which is symmetric about the major and minor axes of the elliptical cross section. The resulting solutions when specialized to flows for circular cross sections agree with those obtained previously by other methods.

It may be emphasized that the procedure in §3 is to seek a representation of the velocity field in powers of cross-sectional variables x_1, x_2 which satisfies the incompressibility conditions, as well as the boundary condition of the vanishing velocity at the pipe wall. Initially, we choose the minimum number of nine terms (see (3.4a, b)) to satisfy these requirements. All subsequent developments in §3 for the constitutive equations and the equations of motion are carried out without approximation and the complete system of equations represents an exact direct theory with nine directors.

2. Theory of directed curves for fluid flow

We summarize here the basic theory in a manner which gives a direct link between the lagrangian and eulerian points of view. Our development in the presence of K directors involves more general representations than those used previously and at the same time includes consideration of thermal effects. Recalling from §1 the description of a model for directed curve $\mathscr{R}_K$, let the curve c in the present configuration at time t be defined by its position vector $\boldsymbol{r}$ relative to a fixed origin, and let θ be a convected (lagrangian) coordinate defining points of the curve. Further, let the K directors be denoted by $\boldsymbol{d}_M$ ($M = 1, 2, \ldots, K$). Then, a motion of the directed curve is specified by

$$\boldsymbol{d}_0 = \boldsymbol{r} = \boldsymbol{r}(\theta, t), \quad \boldsymbol{d}_M = \boldsymbol{d}_M(\theta, t). \tag{2.1}$$

The tangent vector at any point of the curve c and its magnitude are

$$\boldsymbol{a}_3 = \partial \boldsymbol{r}/\partial\theta, \quad a_{33}^{\frac{1}{2}} = (\boldsymbol{a}_3 \cdot \boldsymbol{a}_3)^{\frac{1}{2}} \tag{2.2}$$

and the velocity and director velocities are defined by

$$\boldsymbol{v} = \boldsymbol{w}_0 = \dot{\boldsymbol{r}}, \quad \boldsymbol{w}_M = \dot{\boldsymbol{d}}_M, \tag{2.3}$$

where a superposed dot denotes material time differentiation holding θ fixed.

Let a fixed curve $\bar{c}$ in space be specified by a position vector $\bar{\boldsymbol{r}}$ which is a function of a coordinate ζ on this curve. Then, the tangent vector at any point of $\bar{c}$ and its magnitude are

$$\bar{\boldsymbol{a}}_3 = \partial \bar{\boldsymbol{r}}/\partial\zeta, \quad \bar{a}_{33}^{\frac{1}{2}} = (\bar{\boldsymbol{a}}_3 \cdot \bar{\boldsymbol{a}}_3)^{\frac{1}{2}}. \tag{2.4}$$

The moving curve c (of the directed curves $\mathscr{R}_K$) in its present configuration at time t, coincides with the fixed curve $\bar{c}$, and the velocity of points of the moving curve c at this time is denoted by $\bar{\boldsymbol{v}} = \bar{\boldsymbol{v}}(\zeta, t)$. Further, in the present configuration at time t when c coincides with $\bar{c}$, let the director velocities be denoted by $\bar{\boldsymbol{w}}_M = \bar{\boldsymbol{w}}_M(\zeta, t)$ with $\bar{\boldsymbol{w}}_0 = \bar{\boldsymbol{v}}(\zeta, t)$, and the directors assume the values

$$\bar{\boldsymbol{d}}_M = \bar{\boldsymbol{d}}_M(\zeta)\,(M = 1, 2, \ldots, K), \quad \bar{\boldsymbol{d}}_0 = \bar{\boldsymbol{r}}(\zeta). \tag{2.5}$$

Guided by the developments in Appendix A, we consider an arbitrary part $\alpha \leqslant \theta \leqslant \beta$ of the moving curve c which coincides with the fixed part $\bar{\alpha} \leqslant \zeta \leqslant \bar{\beta}$ of $\bar{c}$ at time t and we postulate both lagrangian and eulerian forms for conservation of mass, momentum, director momentum and moment of momentum for the directed curves as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho y_{MN}\,\mathrm{d}s = \frac{\partial}{\partial t}\int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho}\bar{y}_{MN}\,\mathrm{d}\bar{s} + [\bar{\lambda}\bar{v}_{MN}]_{\bar{\alpha}}^{\bar{\beta}} - \int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho}(\bar{u}_{MN} + \bar{u}^{NM})\,\mathrm{d}\bar{s} = 0 \tag{2.6}$$

for $M, N = 0, 1, 2, \ldots, K$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho \sum_{M=0}^{K} y_{M0}\,\boldsymbol{w}_M\,\mathrm{d}s = \frac{\partial}{\partial t}\int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho} \sum_{M=0}^{K} \bar{y}_{M0}\,\bar{\boldsymbol{w}}_M\,\mathrm{d}\bar{s} + \left[\bar{\lambda}\sum_{M=0}^{K} \bar{v}_{M0}\,\bar{\boldsymbol{w}}_M\right]_{\bar{\alpha}}^{\bar{\beta}} = \int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho}\boldsymbol{f}\,\mathrm{d}\bar{s} + [\boldsymbol{n}]_{\bar{\alpha}}^{\bar{\beta}}, \tag{2.7}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho \sum_{M=0}^{K} y_{MN}\,\boldsymbol{w}_M\,\mathrm{d}s &= \frac{\partial}{\partial t}\int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho} \sum_{M=0}^{K} \bar{y}_{MN}\,\bar{\boldsymbol{w}}_M\,\mathrm{d}\bar{s} + \left[\bar{\lambda}\sum_{M=0}^{K} \bar{v}_{MN}\,\bar{\boldsymbol{w}}_M\right]_{\bar{\alpha}}^{\bar{\beta}} \\ &\quad - \int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho} \sum_{M=0}^{K} \bar{u}_{NM}\,\bar{\boldsymbol{w}}_M\,\mathrm{d}\bar{s} \\ &= \int_{\bar{\alpha}}^{\bar{\beta}} (\bar{\rho}\boldsymbol{l}_N - \boldsymbol{k}_{N3})\,\mathrm{d}\bar{s} + [\boldsymbol{m}_N]_{\bar{\alpha}}^{\bar{\beta}}\end{aligned} \tag{2.8}$$

for $N = 1, 2, \ldots, K$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho \sum_{N=0}^{K}\sum_{M=0}^{K} y_{NM}\,\boldsymbol{d}_N \times \boldsymbol{w}_M\,\mathrm{d}s &= \frac{\partial}{\partial t}\int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho} \sum_{N=0}^{K}\sum_{M=0}^{K} \bar{y}_{NM}\,\bar{\boldsymbol{d}}_N \times \bar{\boldsymbol{w}}_M\,\mathrm{d}\bar{s} \\ &\quad + \left[\bar{\lambda}\sum_{N=0}^{K}\sum_{M=0}^{K} \bar{v}_{NM}\,\bar{\boldsymbol{d}}_N \times \bar{\boldsymbol{w}}_M\right]_{\bar{\alpha}}^{\bar{\beta}} \\ &= \int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho}\sum_{N=0}^{K} \bar{\boldsymbol{d}}_N \times \boldsymbol{l}_N\,\mathrm{d}\bar{s} + \left[\sum_{N=0}^{K} \bar{\boldsymbol{d}}_N \times \boldsymbol{m}_N\right]_{\bar{\alpha}}^{\bar{\beta}},\end{aligned} \tag{2.9}$$

where $\lambda = \rho a_{33}^{\frac{1}{2}}$, $\bar{\lambda} = \bar{\rho}\bar{a}_{33}^{\frac{1}{2}}$, $\mathrm{d}s = a_{33}^{\frac{1}{2}}\,\mathrm{d}\theta$, $\mathrm{d}\bar{s} = \bar{a}_{33}^{\frac{1}{2}}\,\mathrm{d}\zeta$ and $\bar{v}_{NM}$, $\bar{u}_{MN}$, y_{MN}, $\bar{y}_{MN}$ are defined in (A 20) in the appendix. Also $\boldsymbol{n} = \boldsymbol{m}_0$ is the contact force vector, $\boldsymbol{m}_N$ are the contact director force vectors at the curve $\bar{c}$, $\boldsymbol{f} = \boldsymbol{l}_0$ is the assigned force vector, $\boldsymbol{l}_N$ are the assigned director force vectors and $\boldsymbol{k}_{N3}$ are the internal director forces. The force vector $\boldsymbol{f}$ consists of the combined effect of the integrated stress vector on the lateral surface of the jet-like body denoted by $\boldsymbol{f}_c$ (see, for example, the right-hand side of (A 31) of Appendix A) and an integrated contribution arising from the three-dimensional body force denoted by $\boldsymbol{f}_b$ so that $\boldsymbol{f} = \boldsymbol{l}_0 = \boldsymbol{f}_c + \boldsymbol{f}_b$. A parallel statement holds for $\boldsymbol{l}_N$ $(N \geqslant 1)$ so that we may write $\boldsymbol{l}_N = \boldsymbol{l}_{Nc} + \boldsymbol{l}_{Nb}$.

Balance of entropy† and energy for every part $\alpha \leqslant \theta \leqslant \beta$ of c which coincides with the part $\bar{\alpha} \leqslant \zeta \leqslant \bar{\beta}$ of the fixed curve $\bar{c}$ at time t are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho\tilde{\eta}_N\,\mathrm{d}s &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \rho \sum_{M=0}^{K} y_{MN}\,\eta_M\,\mathrm{d}s \\ &= \frac{\partial}{\partial t}\int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho}\sum_{M=0}^{K}\bar{y}_{MN}\,\bar{\eta}_M\,\mathrm{d}\bar{s} + \left[\bar{\lambda}\sum_{M=0}^{K}\bar{v}_{MN}\,\bar{\eta}_M\right]_{\bar{\alpha}}^{\bar{\beta}} - \int_{\bar{\alpha}}^{\bar{\beta}}\bar{\rho}\sum_{M=0}^{K}\bar{u}_{NM}\,\bar{\eta}_M\,\mathrm{d}\bar{s} \\ &= \int_{\bar{\alpha}}^{\bar{\beta}}\bar{\rho}(s_N+\xi_N)\,\mathrm{d}\bar{s} - [k_N]_{\bar{\alpha}}^{\bar{\beta}} \quad (N = 0, 1, \ldots, K)\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta\left(\epsilon + \frac{1}{2}\sum_{M=0}^{K}\sum_{N=0}^{K} y_{MN}\,\boldsymbol{w}_M\cdot\boldsymbol{w}_N\right)\rho\,\mathrm{d}s &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta\sum_{M=0}^{K}\sum_{N=0}^{K} y_{MN}(\epsilon_{MN}+\tfrac{1}{2}\boldsymbol{w}_M\cdot\boldsymbol{w}_N)\,\rho\,\mathrm{d}s \\ &= \frac{\partial}{\partial t}\int_{\bar{\alpha}}^{\bar{\beta}}\sum_{M=0}^{K}\sum_{N=0}^{K}\bar{y}_{MN}(\bar{\epsilon}_{MN}+\tfrac{1}{2}\bar{\boldsymbol{w}}_M\cdot\bar{\boldsymbol{w}}_N)\,\bar{\rho}\,\mathrm{d}s \\ &\quad + \left[\bar{\lambda}\sum_{M=0}^{K}\sum_{N=0}^{K}\bar{v}_{MN}(\bar{\epsilon}_{MN}+\tfrac{1}{2}\bar{\boldsymbol{w}}_M\cdot\bar{\boldsymbol{w}}_N)\right]_{\bar{\alpha}}^{\bar{\beta}} \\ &= \int_{\bar{\alpha}}^{\bar{\beta}}\bar{\rho}\sum_{N=0}^{K}(r_N+\boldsymbol{l}_N\cdot\bar{\boldsymbol{w}}_N)\,\mathrm{d}\bar{s} \\ &\quad + \left[\sum_{N=0}^{K}(\boldsymbol{m}_N\cdot\bar{\boldsymbol{w}}_N - h_N)\right]_{\bar{\alpha}}^{\bar{\beta}}.\end{aligned} \tag{2.11}$$

In these equations (of the direct theory) $\tilde{\eta}_N$, $\eta_N(\theta,t) = \bar{\eta}_N(\zeta,t)$ are the entropy densities, k_N are the entropy fluxes, h_N are heat fluxes, s_N are the specific external rates of supply of entropy, r_N are the specific external rates of supply of heat, $\bar{\epsilon}$, $\epsilon_{MN}(\theta,t) = \bar{\epsilon}_{MN}(\zeta,t)$ are the specific internal energies. All these entities, if desired, can be interpreted in accordance with various formulae in Appendix A (see, for example, equations (A 24) and (A 25)). Further, let θ_N $(N = 0, 1, 2, \ldots, K)$ represent the effects of temperature in the rod-like or tube-like body. Then, the external rates of supply of heat and heat fluxes can be related to external supply of entropy and entropy fluxes (Green & Naghdi 1977, 1979) by

$$r_N = \theta_N s_N, \quad h_N = \theta_N k_N \quad (N = 0, 1, \ldots, K\text{: not summed}), \tag{2.12}$$

respectively. (The notation for the convected coordinate θ introduced in (2.1) can be easily distinguished from θ_N for the temperatures in (2.12) and elsewhere in this section by the fact that the latter always involves a value for the index N.)

† Balance of entropy of the type used here (within the scope of three-dimensional theory) was first introduced by Green & Naghdi (1977) and subsequently utilized in different contexts (see, for example, Green & Naghdi 1979, 1985, 1987).

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The local field equations corresponding to (2.6) are

$$\dot{\rho}+\rho \boldsymbol{a}^3\cdot\partial\boldsymbol{v}/\partial\theta=0 \quad \text{or} \quad \rho a_{33}^{\frac{1}{2}}=\lambda(\theta), \quad y_{MN}=\text{function of } \theta, \tag{2.13a}$$

or

$$\frac{\partial(\bar{\lambda}\bar{y}_{MN})}{\partial t}+\frac{\partial(\bar{\lambda}\bar{v}_{MN})}{\partial\zeta}-\bar{\lambda}(\bar{u}_{MN}+\bar{u}_{NM})=0. \tag{2.13b}$$

Similarly, with the help of (2.13*b*), the local field equations corresponding to (2.7) to (2.9) are

$$\lambda\sum_{M=0}^{K} y_{MN}\dot{\boldsymbol{w}}_M=\bar{\lambda}\sum_{M=0}^{K}\left\{\bar{y}_{MN}\frac{\partial\bar{\boldsymbol{w}}_M}{\partial t}+\bar{v}_{MN}\frac{\partial\bar{\boldsymbol{w}}_M}{\partial\zeta}+\bar{u}_{MN}\bar{\boldsymbol{w}}_M\right\}=\partial\boldsymbol{m}_N/\partial\zeta+\lambda\boldsymbol{l}_N-\boldsymbol{k}_N \tag{2.14}$$

for $N=0,1,\ldots,K$, with $\boldsymbol{k}_0=\boldsymbol{0}$, $\boldsymbol{m}_0=\boldsymbol{n}$, $\boldsymbol{k}_N=\boldsymbol{k}_{N3}\bar{a}_{33}^{\frac{1}{2}}$ and

$$\sum_{N=0}^{K}\left(\bar{\boldsymbol{d}}_N\times\boldsymbol{k}_N+\frac{\partial\bar{\boldsymbol{d}}_N}{\partial\zeta}\times\boldsymbol{m}_N\right)=\boldsymbol{0}, \tag{2.15}$$

where in obtaining (2.15) from (2.9) we have made use of the identity

$$\sum_{M=0}^{K}\sum_{N=0}^{K}\left(\bar{v}_{MN}\frac{\partial\bar{\boldsymbol{d}}_M}{\partial\zeta}+\bar{u}_{MN}\bar{\boldsymbol{d}}_M\right)\times\bar{\boldsymbol{w}}_N=\boldsymbol{0}. \tag{2.16}$$

Again, using (2.13) and (2.14), the local field equations corresponding to (2.10) and (2.11) are

$$\lambda\dot{\tilde{\eta}}_N=\lambda\sum_{M=0}^{K} y_{MN}\dot{\eta}_M=\bar{\lambda}\sum_{M=0}^{K}\left\{\bar{y}_{MN}\frac{\partial\bar{\eta}_M}{\partial t}+\bar{v}_{MN}\frac{\partial\bar{\eta}_M}{\partial t}+\bar{u}_{MN}\bar{\eta}_M\right\}=\bar{\lambda}(s_N+\xi_N)-\partial k_N/\partial\zeta, \tag{2.17}$$

$$\begin{aligned}\lambda\dot{\epsilon}&=\lambda\sum_{M=0}^{K}\sum_{N=0}^{K} y_{MN}\dot{\epsilon}_{MN}\\&=\bar{\lambda}\sum_{M=0}^{K}\sum_{N=0}^{K}\left\{\bar{y}_{MN}\frac{\partial\bar{\epsilon}_{MN}}{\partial t}+\bar{v}_{MN}\frac{\partial\bar{\epsilon}_{MN}}{\partial\zeta}+(\bar{u}_{MN}+\bar{u}_{NM})\bar{\epsilon}_{MN}\right\}\\&=\sum_{N=0}^{K}\left(\bar{\lambda}\theta_N s_N-k_N\frac{\partial\theta_N}{\partial\zeta}\right)+P-\sum_{N=0}^{K}\theta_N\frac{\partial k_N}{\partial\zeta},\end{aligned} \tag{2.18}$$

where the mechanical power P is defined by

$$P=\sum_{N=0}^{K}\left(\boldsymbol{k}_N\cdot\bar{\boldsymbol{w}}_N+\boldsymbol{m}_N\cdot\frac{\partial\bar{\boldsymbol{w}}_N}{\partial\zeta}\right), \tag{2.19}$$

and external body forces are eliminated with the help of (2.14). After elimination of external rates of supply of heat using (2.17), equation (2.18) becomes

$$\begin{aligned}\lambda\left[\dot{\psi}+\sum_{N=0}^{K}\tilde{\eta}_N\dot{\theta}_N\right]&=\lambda\sum_{M=0}^{K}\sum_{N=0}^{K} y_{MN}(\dot{\psi}_{MN}+\eta_M\dot{\theta}_N)\\&=\bar{\lambda}\sum_{M=0}^{K}\sum_{N=0}^{K}\left\{\bar{y}_{MN}\left(\frac{\partial\bar{\psi}_{MN}}{\partial t}+\bar{\eta}_M\frac{\partial\theta_N}{\partial t}\right)+\bar{v}_{MN}\left(\frac{\partial\bar{\psi}_{MN}}{\partial\zeta}+\bar{\eta}_M\frac{\partial\theta_N}{\partial\zeta}\right)\right.\\&\qquad\left.+\bar{u}_{MN}(2\bar{\psi}_{MN}+\bar{\eta}_N\theta_M)\right\}\\&=P-\sum_{N=0}^{K}\left(\bar{\lambda}\theta_N\xi_N+k_N\frac{\partial\theta_N}{\partial\zeta}\right),\end{aligned} \tag{2.20}$$

where
$$\left.\begin{aligned}\psi &= \epsilon - \sum_{N=0}^{K} \theta_N \tilde{\eta}_N = \sum_{M=0}^{K} \sum_{N=0}^{K} y_{MN}(\epsilon_{MN} - \theta_N \eta_M),\\ \bar{\psi}_{MN} &= \bar{\epsilon}_{MN} - \tfrac{1}{2}(\bar{\eta}_M \theta_N + \bar{\eta}_N \theta_M).\end{aligned}\right\} \quad (2.21)$$

After constitutive equations are introduced for the response functions

$$\psi(\text{or } \psi_{MN}, \bar{\psi}_{MN}),\ \tilde{\eta}_N(\text{or } \eta_N, \bar{\eta}_N),\ \theta_N,\ \xi_N,\ k_N,\ \boldsymbol{k}_N,\ \boldsymbol{m}_N, \quad (2.22)$$

then the reduced energy equation (2.20) is regarded as an identity to be satisfied for all thermomechanical processes.

For a complete theory we need to examine restrictions arising from interpretations of the Second Law of thermodynamics, but we omit this here.

It is important to provide here some additional remarks pertaining to certain characteristic features of the direct theory and the nature of its predictive capability. We confine our remarks to only the isothermal (or mechanical) theory of this section whose basic field equations consist of the conservation of mass equation (2.13*a*) or (2.13*b*), the momentum and director momenta equations (2.14) and the moment of momentum equation (2.15), although a parallel discussion holds also in the presence of thermal effects. First, several important features of a direct theory of the type developed here should be noted: (1) by the nature of its construction, invariance under superposed rigid body motions is automatically satisfied at each order of the hierarchy in the general theory†; (2) closure exists at every hierarchical order; and (3) the theory provides an attractive avenue for obtaining analytical solutions, even though at first sight the complex structure and length of the basic equations may be somewhat discouraging.

In the course of utilization of the basic equations of the direct theory, sometimes it becomes desirable to restrict some of the kinematical variables (or some of their components) by imposing suitable constraints. (An example of such a constraint is, of course, the well-known incompressibility condition which in the context of the exact three-dimensional theory induces the presence of a Lagrange multiplier as the pressure term when the Navier–Stokes equations are specialized to the case of an incompressible fluid.) With reference to the direct theory of this section, frequently it is of interest either from considerations of the physical aspects of a given problem or for computational convenience to introduce several constraints. For clarity's sake and later reference, we indicate briefly how such constraints result in a number of Lagrange multipliers in the complete theory which include also the appropriate constitutive equations. (For a more detailed account of constraints in the context of directed curve, see Naghdi (1982, §12).) To this end, consider a class of constraints which are linear relations between the kinematical variables of the type

$$\bar{\boldsymbol{w}}_{0,\zeta} = \bar{\boldsymbol{v}}_{,\zeta},\ \bar{\boldsymbol{w}}_N,\ \bar{\boldsymbol{w}}_{N,\zeta} \quad (2.23)$$

for some values of N. Then, it is assumed that each of the functions $\boldsymbol{m}_0 = \boldsymbol{n}, \boldsymbol{m}_N, \boldsymbol{k}_N$ are determined to within an additive constraint response so that each kinetical function such as $\boldsymbol{n}$ can be written as

$$(\ ) = (\ )_{\text{ind}} + (\ )_{\text{det}}, \quad (2.24)$$

where the determinate parts $(\ )_{\text{det}}$ are specified by constitutive equations and the

† A general definition for the order of hierarchies appropriate for the theory of Cosserat (or directed) curve is given in Naghdi (1982, §9). For example, for the lowest order of hierarchy the theory uses only two directors and a space curve.

indeterminate parts $(\ )_{\text{ind}}$ are arbitrary functions of ζ, t and are workless. Recalling (2.19), the expression representing the worklessness of the indeterminate parts is given by

$$\Sigma\,(\boldsymbol{k}_N)_{\text{ind}}\cdot\overline{\boldsymbol{w}}_N+(\boldsymbol{m}_N)_{\text{ind}}\cdot\partial\overline{\boldsymbol{w}}_N/\partial\zeta=0, \tag{2.25}$$

where the summation is intended only over those values of N which occur in the constraint equations. The indeterminate parts of the kinetical quantities are then determined from (2.25) and the constraint equations in terms of Lagrange multipliers, as will become evident in the early part of the next section.

3. Flow of viscous fluid in an elliptical pipe with varying cross sections

We apply the mechanical part of the theory of §2 to problems of flow of an incompressible linear viscous fluid of constant density ρ^* and constant viscosity coefficient μ in a pipe of variable cross section and with rigid wall, bounded by the fixed surface

$$x_1^2/a^2+x_2^2/b^2=1, \tag{3.1}$$

where a and b are the major and minor axes of the elliptical cross section and depend on $x_3=\zeta$. The pipe and the motion of the fluid are referred to a rectangular cartesian coordinate system $x_i\,(i=1,2,3)$ with constant orthonormal basis $\boldsymbol{e}_i$ such that the axis of the pipe is the line $x_1=0$, $x_2=0$. In the theory of §2 we choose $\zeta^i=x_i$ with the fixed curve $\bar{c}$ coinciding with the x_3-axis and then from (2.4) we have

$$\bar{\boldsymbol{r}}=\zeta\boldsymbol{e}_3,\quad \zeta=x_3,\quad \boldsymbol{a}_3=\boldsymbol{e}_3,\quad a_{33}=1. \tag{3.2}$$

Referred to the basis $\boldsymbol{e}_i$, the velocity vector $\bar{\boldsymbol{v}}$, the director velocity vectors $\overline{\boldsymbol{w}}_N$ and the various kinetical quantities in (2.13)–(2.15) can be expressed as

$$\left.\begin{aligned}\bar{\boldsymbol{v}}=v_i\boldsymbol{e}_i,\quad \overline{\boldsymbol{w}}_N=w_{Ni}\boldsymbol{e}_i,\quad w_{0i}=v_i,\quad \boldsymbol{n}=n_i\boldsymbol{e}_i,\\ \boldsymbol{m}_N=m_{Ni}\boldsymbol{e}_i,\quad \boldsymbol{k}_N=k_{Ni}\boldsymbol{e}_i,\quad \boldsymbol{f}=f_i\boldsymbol{e}_i,\quad \boldsymbol{l}_N=l_{Ni}\boldsymbol{e}_i,\end{aligned}\right\} \tag{3.3}$$

where the usual rectangular cartesian tensor notation is adopted in (3.3) and elsewhere in this paper, together with summation convention over repeated lower case Latin indices.

Our aim in what follows is to select the lowest hierarchy which satisfies not only the incompressibility condition but also the kinematical boundary condition on the lateral surface of the fluid in the pipe. We recall for clarity that in problems of free-surface jets with cross section in the form (3.1) it was possible to obtain a good representation for the flow velocity with only two directors, since no restriction was imposed on the velocity at the free surface of the jet and the two directors were sufficient to satisfy the condition of incompressibility. However, in the case of fluid flow inside a rigid pipe with cross section (3.1), we need a polynomial representation containing at least up to cubic terms in x_1 and x_2 for an adequate representation of the flow or equivalently the velocity $\boldsymbol{v}^*$ (see (A 11)). Such a representation for $\boldsymbol{v}^*$ corresponds to the use of nine directors in the theory of §2 and also motivates the choice of $\bar{\lambda}_i$.

Keeping the above background in mind, from §2 we select a theory of a directed curve in which the representation (A 11) for the three-dimensional velocity $\boldsymbol{v}^*$ has polynomial weighting functions

$$\left.\begin{aligned}\bar{\lambda}_1=x_1,\quad \bar{\lambda}_2=x_2,\quad \bar{\lambda}_3=x_1^2,\quad \bar{\lambda}_4=x_1x_2,\quad \bar{\lambda}_5=x_2^2,\\ \bar{\lambda}_6=x_1^3,\quad \bar{\lambda}_7=x_1^2x_2,\quad \bar{\lambda}_8=x_1x_2^2,\quad \bar{\lambda}_9=x_2^3.\end{aligned}\right\} \tag{3.4a}$$

Also, $$\bar{\boldsymbol{d}}_0 = \zeta \boldsymbol{e}_3, \quad \boldsymbol{d}_1 = \boldsymbol{e}_1, \quad \boldsymbol{d}_2 = \boldsymbol{e}_2, \quad \boldsymbol{d}_N = \boldsymbol{0} \quad (N \geqslant 10). \tag{3.4b}$$

The condition of incompressibility (in the three-dimensional theory), namely $\boldsymbol{v}^*_{,i} \cdot \boldsymbol{e}_i = 0$, after using the representation (A 11) together with (3.4) implies the restrictions

$$\left.\begin{aligned} w_{11}+w_{22}+v_3' = 0, \quad 2w_{31}+w_{42}+w_{13}' = 0,\\ w_{41}+2w_{52}+w_{23}' = 0, \quad 3w_{61}+w_{72}+w_{33}' = 0,\\ 2w_{71}+2w_{82}+w_{43}' = 0, \quad w_{81}+3w_{92}+w_{53}' = 0, \end{aligned}\right\} \tag{3.5}$$

where a prime denotes differentiation with respect to ζ. In addition, we impose the constraints

$$w_{63} = 0, \quad w_{73} = 0, \quad w_{83} = 0, \quad w_{93} = 0 \tag{3.6}$$

on the last four directors. (As in Appendix A, symbols with added asterisks (such as $\boldsymbol{v}^*$) are used to designate quantities in the three-dimensional theory in order to avoid confusion with corresponding symbols such as $\boldsymbol{v}$ (see (2.3) employed in direct theory.) It follows from the discussion pertaining to constraints in the last paragraph of §2 that the conditions (3.5) and (3.6) give rise to constraint responses which permit the various kinetical quantities $\boldsymbol{n}, \boldsymbol{m}_N, \boldsymbol{k}_N$ to be expressed as (see Naghdi 1982):

$$\left.\begin{aligned} &\boldsymbol{n} = -p\boldsymbol{e}_3+\hat{\boldsymbol{n}}, \quad \boldsymbol{m}_1 = -p_1\boldsymbol{e}_3+\hat{\boldsymbol{m}}_1, \quad \boldsymbol{m}_2 = -p_2\boldsymbol{e}_3+\hat{\boldsymbol{m}}_2,\\ &\boldsymbol{m}_3 = -p_3\boldsymbol{e}_3+\hat{\boldsymbol{m}}_3, \quad \boldsymbol{m}_4 = -p_4\boldsymbol{e}_3+\hat{\boldsymbol{m}}_4, \quad \boldsymbol{m}_5 = -p_5\boldsymbol{e}_3+\hat{\boldsymbol{m}}_5,\\ &\boldsymbol{m}_6 = -q_6\boldsymbol{e}_3+\hat{\boldsymbol{m}}_6, \quad \boldsymbol{m}_7 = -q_7\boldsymbol{e}_3+\hat{\boldsymbol{m}}_7, \quad \boldsymbol{m}_8 = -q_8\boldsymbol{e}_3+\hat{\boldsymbol{m}}_8,\\ &\boldsymbol{m}_9 = -q_9\boldsymbol{e}_3+\hat{\boldsymbol{m}}_9, \quad \boldsymbol{k}_1 = -p\boldsymbol{e}_1+\hat{\boldsymbol{k}}_1, \quad \boldsymbol{k}_2 = -p\boldsymbol{e}_2+\hat{\boldsymbol{k}}_2,\\ &\boldsymbol{k}_3 = -2p_1\boldsymbol{e}_1+\hat{\boldsymbol{k}}_3, \quad \boldsymbol{k}_4 = -p_2\boldsymbol{e}_1-p_1\boldsymbol{e}_2+\hat{\boldsymbol{k}}_4, \quad \boldsymbol{k}_5 = -2p_2\boldsymbol{e}_2+\hat{\boldsymbol{k}}_5,\\ &\boldsymbol{k}_6 = -3p_3\boldsymbol{e}_1+r_6\boldsymbol{e}_3+\hat{\boldsymbol{k}}_6, \quad \boldsymbol{k}_7 = -2p_4\boldsymbol{e}_1-p_3\boldsymbol{e}_2+r_7\boldsymbol{e}_3+\hat{\boldsymbol{k}}_7,\\ &\boldsymbol{k}_8 = -p_5\boldsymbol{e}_1-2p_4\boldsymbol{e}_2+r_8\boldsymbol{e}_3+\hat{\boldsymbol{k}}_8, \quad \boldsymbol{k}_9 = -3p_5\boldsymbol{e}_2+r_9\boldsymbol{e}_3+\hat{\boldsymbol{k}}_9, \end{aligned}\right\} \tag{3.7a}$$

where $p_1, \ldots, p_5$, $r_6, \ldots, r_9$, $q_6, \ldots, q_9$ are arbitrary scalar functions of ζ, t and $\hat{\boldsymbol{m}}_N$, $\hat{\boldsymbol{k}}_N$ are specified by constitutive equations. In view of (3.4b), equation (2.15) is satisfied if

$$\boldsymbol{e}_1 \times \hat{\boldsymbol{k}}_1 + \boldsymbol{e}_2 \times \hat{\boldsymbol{k}}_2 + \boldsymbol{e}_3 \times \hat{\boldsymbol{n}} = \boldsymbol{0}. \tag{3.7b}$$

Assuming that the viscous fluid remains in contact with the fixed surface (3.1), it follows with the help of the representation (A 11) for the velocity field $\boldsymbol{v}^*$ that

$$\boldsymbol{v} = -b^2\boldsymbol{w}_5 = -a^2\boldsymbol{w}_3, \quad \boldsymbol{w}_4 = \boldsymbol{0}, \quad \boldsymbol{w}_1 = -a^2\boldsymbol{w}_6 = -b^2\boldsymbol{w}_8, \quad \boldsymbol{w}_2 = -a^2\boldsymbol{w}_7 = -b^2\boldsymbol{w}_9. \tag{3.8}$$

From (3.5) and (3.8) it is found that the non-zero components of the velocity and director velocities may be expressed in the form

$$\left.\begin{aligned} &v_3 = A/(ab), \quad w_{11} = Aa'/(a^2b), \quad w_{12} = -B/a^2,\\ &w_{21} = B/b^2, \quad w_{22} = Ab'/(ab^2), \quad w_{33} = -A/(a^3b),\\ &w_{53} = -A/(ab^3), \quad w_{61} = -Aa'/(a^4b), \quad w_{62} = B/a^4,\\ &w_{71} = -B/(a^2b^2), \quad w_{72} = -Ab'/(a^3b^2), \quad w_{81} = -Aa'/(a^2b^3),\\ &w_{82} = B/(a^2b^2), \quad w_{91} = -B/b^4, \quad w_{92} = -Ab'/(ab^4), \end{aligned}\right\} \tag{3.9}$$

where $A = A(t)$, $B = B(\zeta, t)$ and prime was defined earlier following (3.5). The results (3.9) involve a cross flow in sections of the pipe (or tube) due to variations of a, b with

ζ and the function $B(\zeta, t)$ is associated with cross flows in each section of the pipe representing swirling motions about the axis of the pipe. This swirling motion is easily seen from the expression for $\boldsymbol{v}^*$ (in the three-dimensional theory), namely

$$\boldsymbol{v}^* = \boldsymbol{e}_1\left(1-\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}\right)\left(\frac{Aa'x_1}{a^2b}+\frac{Bx_2}{b^2}\right)+\boldsymbol{e}_2\left(1-\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}\right)\left(-\frac{Bx_1}{a^2}+\frac{Ab'x_2}{ab^2}\right) + \boldsymbol{e}_3\frac{A}{ab}\left(1-\frac{x_1^2}{a^2}-\frac{x_2^2}{b^2}\right),$$

which is recorded with the help of (A 7) and (3.9). Also, the integrated normal velocity across a pipe cross section, which represents the rate of volume flow Q of fluid down the pipe is given by

$$Q = \tfrac{1}{2}\pi A(t) \tag{3.10}$$

and is independent of ζ.

For an incompressible linear viscous fluid we assume the response functions

$$\boldsymbol{n}, \hat{\boldsymbol{m}}_N, \hat{\boldsymbol{k}}_N \tag{3.11}$$

are linearly dependent on velocity gradients, director velocity gradients and director velocities. Values for the coefficients in these constitutive equations may be selected by comparison with three-dimensional forms with the help of (A 11) and (A 30). Details of this calculation are omitted, as well as the relevant values of $\bar{y}_{MN}$, $\bar{v}_{MN}$, $\bar{u}_{MN}$ which may be computed with the help of (A 20) and (A 35). Leaving aside those equations of motion which involve the constraint response functions $q_6, \ldots, q_9, r_6, \ldots, r_9$, since they provide relations between the constraint response functions and are not needed in the determination of the quantities of interest, the relevant final equations of motion from (2.14) are

$$\left.\begin{aligned} f_1 = 0, \quad f_2 = 0, \quad \lambda f_3 - p' &= \tfrac{1}{2}\rho^*\pi\dot{A} + \tfrac{1}{3}\rho^*\pi A^2(1/ab)', \\ \lambda l_{11} + p &= -\tfrac{1}{24}\mu\pi A(a^2)''' + \tfrac{1}{24}\rho^*\pi A(a^2)' + \tfrac{1}{24}\rho^*\pi A^2 a(a'/ab)' - \rho^*\pi B^2 a/24b, \\ \lambda l_{12} &= \tfrac{1}{12}\mu\pi(Bab)'' - \tfrac{1}{12}\rho^*\pi\dot{B}ab - (\rho^*\pi Aa/24b)Bb/a)', \quad \lambda l_{13} - p_1' = 0, \end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} \lambda l_{21} &= -\tfrac{1}{12}\mu\pi(Bab)'' + \tfrac{1}{12}\rho^*\pi\dot{B}ab + (\rho^*\pi Ab/24a)(Ba/b)', \\ \lambda l_{22} + p &= -\tfrac{1}{24}\mu\pi A(b^2)''' + \tfrac{1}{24}\rho^*\pi\dot{A}(b^2)' + \tfrac{1}{24}\rho^*\pi A^2 b(b'/ab)' - \rho^*\pi B^2 b/24a, \\ \lambda l_{23} - p_2' &= 0, \quad \lambda l_{31} + 2p_1 = 0, \quad l_{32} = 0, \\ \lambda l_{33} - p_3' &= -\mu\pi A - \tfrac{1}{12}\mu\pi A(a^2)'' + \tfrac{1}{12}\rho^*\pi\dot{A}a^2 - \rho^*\pi A^2(ab)'/24b^2, \\ l_{51} &= 0, \quad \lambda l_{52} + 2p_2 = 0, \\ \lambda l_{53} - p_5' &= -\mu\pi A - \tfrac{1}{12}\mu\pi A(b^2)'' + \tfrac{1}{12}\rho^*\pi\dot{A}b^2 - \rho^*\pi A^2(ab)'/24a^2, \\ \lambda l_{41} + p_2 &= 0, \quad \lambda l_{42} + p_1 = 0, \quad \lambda l_{43} + p_4' = 0, \\ \lambda l_{61} + 3p_3 &= -\tfrac{1}{4}\mu\pi A(a^2)' - \tfrac{1}{128}\mu\pi A(a^4)''' + \tfrac{1}{128}\rho^*\pi\dot{A}(a^4)' \\ &\quad + \tfrac{1}{80}\rho^*\pi A^2 a^3(a'/ab)' - \rho^*\pi B^2 a^3/80b, \\ \lambda l_{62} &= \tfrac{1}{2}\mu\pi Bab + \tfrac{1}{32}\mu\pi(Ba^2b)'' - \tfrac{1}{32}\rho^*\pi\dot{B}a^3b - (\rho^*\pi Aa^3/80b)(Bb/a)', \end{aligned}\right\} \tag{3.13}$$

and

$$\left.\begin{aligned}
\lambda l_{71}+2p_4 &= -\tfrac{1}{6}\mu\pi Bab-\tfrac{1}{96}\mu\pi(Ba^3b)''+\tfrac{1}{96}\rho^*\pi\dot{B}a^3b+\tfrac{1}{240}\rho^*\pi Aab(Ba/b)',\\
\lambda l_{72}+p_3 &= -\tfrac{1}{12}\mu\pi A(b^2)'-\tfrac{1}{96}\mu\pi A(a^2bb')''+\tfrac{1}{96}\rho^*\pi\dot{A}a^2bb'\\
&\quad+\tfrac{1}{240}\rho^*\pi A^2a^2b(b'/ab)'-\tfrac{1}{240}\rho^*\pi B^2ab,\\
\lambda l_{81}+p_5 &= -\tfrac{1}{12}\mu\pi A(a^2)'-\tfrac{1}{96}\mu\pi A(ab^2a')''+\tfrac{1}{96}\rho^*\pi\dot{A}ab^2a'\\
&\quad+\tfrac{1}{240}\rho^*\pi A^2ab^2(a'/ab)'-\tfrac{1}{240}\rho^*\pi B^2ab,\\
\lambda l_{82}+2p_4 &= \tfrac{1}{6}\mu\pi Bab+\tfrac{1}{96}\mu\pi(Bab^3)''-\tfrac{1}{96}\rho^*\pi\dot{B}ab^3-\tfrac{1}{240}\rho^*\pi Aab(Bb/a)',\\
\lambda l_{91} &= -\tfrac{1}{2}\mu\pi Bab-\tfrac{1}{32}\mu\pi(Bab^3)''+\tfrac{1}{32}\rho^*\pi\dot{B}ab^3+(\rho^*\pi Ab^3/80a)(Ba/b)',\\
\lambda l_{92}+3p_5 &= -\tfrac{1}{4}\mu\pi A(b^2)'-\tfrac{1}{128}\mu\pi A(b^4)'''+\tfrac{1}{128}\rho^*\pi\dot{A}(b^4)'\\
&\quad+\tfrac{1}{80}\rho^*\pi A^3b^3(b'/ab)'-\rho^*\pi B^2b^3/80a,
\end{aligned}\right\} \quad (3.14)$$

where a superposed dot denotes partial differentiation with respect to t. In the special case when the cross section of the pipe is circular, so that $b = a$, the equations (3.12) to (3.14) are equivalent to those derived by Caulk & Naghdi (1987) from the three-dimensional equations for an incompressible viscous fluid.

In the absence of body forces, it follows from (A 31) that for the elliptical pipe whose boundary is specified by (3.1)

$$-\boldsymbol{f}+\frac{\boldsymbol{l}_3}{a^2}+\frac{\boldsymbol{l}_5}{b^2}=\mathbf{0},\quad -\boldsymbol{l}_1+\frac{\boldsymbol{l}_6}{a^2}+\frac{\boldsymbol{l}_8}{b^2}=\mathbf{0},\quad -\boldsymbol{l}_2+\frac{\boldsymbol{l}_7}{a^2}+\frac{\boldsymbol{l}_9}{b^2}=\mathbf{0}. \quad (3.15)$$

With the help of (3.15), surface tractions may be eliminated from equations (3.12) to (3.14) to yield the following seven equations for p, p_1, p_2, p_3, p_4, p_5, B, namely

$$p_1 = p_2 = 0, \quad (3.16)$$

$$\begin{aligned}
-p'+(p_3')/a^2+(p_5')/b^2 &= X\\
&= \mu\pi A\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\tfrac{1}{12}\mu\pi A\left\{\frac{(a^2)''}{a^2}+\frac{(b^2)''}{b^2}\right\}+\tfrac{1}{3}\rho^*\pi\dot{A}-\frac{\rho^*\pi A^2(ab)'}{4a^2b^2},
\end{aligned} \quad (3.17)$$

$$\begin{aligned}
p-3p_3/a^2-p_5/b^2 = Y &= \mu\pi Aaa'(1/2a^2+1/6b^2)-\tfrac{1}{24}\mu\pi A(a^2)'''+\mu\pi A\{(a^4)'''/128a^2\\
&\quad+(ab^2a')''/96b^2\}+\tfrac{1}{48}\rho^*\pi\dot{A}(a^2)'+\tfrac{1}{40}\rho^*\pi A^2a(a'/ab)'\\
&\quad-\rho^*\pi B^2a/40b,
\end{aligned} \quad (3.18)$$

$$\begin{aligned}
p-p_3/a^2-3p_5/b^2 &= Z\\
&= \mu\pi Abb'(1/6a^2+1/2b^2)-\tfrac{1}{24}\mu\pi A(b^2)'''+\mu\pi A\{(b^4)'''/128b^2\\
&\quad+(a^2bb')''/96a^2\}+\tfrac{1}{48}\rho^*\pi\dot{A}(b^2)'+\tfrac{1}{40}\rho^*\pi A^2b(b'/ab)'\\
&\quad-\rho^*\pi B^2b/40a,
\end{aligned} \quad (3.19)$$

$$\begin{aligned}
\frac{2p_4}{b^2} &= \mu\pi B\left(\frac{b}{2a}+\frac{a}{6b}\right)-\mu\pi\left\{\tfrac{1}{12}(Bab)''-\frac{(Ba^3b)''}{32a^2}-\frac{(Bab^3)''}{96b^2}\right\}\\
&\quad+\tfrac{1}{24}\rho^*\pi\dot{B}ab+(\rho^*\pi Aa/40b)(Bb/a)',
\end{aligned} \quad (3.20)$$

$$\begin{aligned}
\frac{2p_4}{a^2} &= -\mu\pi B\left(\frac{b}{6a}+\frac{a}{2b}\right)+\mu\pi\left\{\tfrac{1}{12}(Bab)''-\frac{(Bab^3)''}{32b^2}-\frac{(Ba^3b)''}{96a^2}\right\}\\
&\quad-\tfrac{1}{24}\rho^*\pi\dot{B}ab-(\rho^*\pi Ab/40a)(Ba/b)'.
\end{aligned} \quad (3.21)$$

The functions p_3, p_4, p_5 may be eliminated from (3.17) to (3.21) to yield two equations for p and B, namely

$$-\tfrac{1}{2}ab(p/ab)' = X+\tfrac{1}{4}(Y'+Z)+\tfrac{1}{8}(3Y-Z)(a^2)'/a^2-\tfrac{1}{8}(Y-3Z)(b^2)'/b^2, \qquad (3.22)$$

and

$$\mu B(3a^4+2a^2b^2+3b^4)/(6ab)-\tfrac{1}{12}\mu(a^2+b^2)(Bab)''+\mu(a^2+3b^2)(Ba^3b)''/(96a^2) + \mu(3a^2+b^2)(Bab^3)''/(96b^2)+\tfrac{1}{24}\rho^*ab(a^2+b^2)\dot{B}+\rho^*abA[(Ba/b)'+\tfrac{1}{40}(Bb/a)'] = 0, \qquad (3.23)$$

where the quantities X, Y, Z which occur in (3.22) are defined in (3.17)–(3.19).

We consider two applications of the theory in this section. In the first a swirling motion ($B \neq 0$) is superposed on uniform axial flow in a uniform elliptical pipe with constant semi-axes a, b. With A a constant and remembering that by $(3.9)_1$ the quantity $A/(ab)$ is a constant velocity, equation (3.23) reduces to

$$abB''-\tfrac{3}{5}R^*(ab)^{\frac{1}{2}}B'-4(3a^4+2a^2b^2+3b^4)B/\{ab(a^2+b^2)\}-\rho^*ab\dot{B}/\mu = 0, \qquad (3.24)$$

where the Reynolds number R^* is given by

$$R^* = \rho^*A/[\mu(ab)^{\frac{1}{2}}]. \qquad (3.25)$$

Also, equation (3.22) becomes

$$-p' = 2[(a^2+b^2)/(a^2b^2)][\mu A-\rho^*abBB'/(80)] \qquad (3.26)$$

which may be integrated to yield

$$-p = 2[(a^2+b^2)/(a^2b^2)][\mu A\zeta-\rho^*abB^2/(160)]+K(t), \qquad (3.27)$$

where $K(t)$ is a function of t. A relevant steady-state solution of (3.24) gives a value for B which represents exponential decay as we move along the pipe. This result agrees with that of Caulk & Naghdi (1987) when the pipe is circular.

The second example is of flows which are symmetric about the major and minor axes of the elliptical cross section of the pipe so that $B = 0$, and hence from (3.20) and (3.21) the scalar $p_4 = 0$ and the relevant equation is (3.22). Consider an infinite pipe which is such that

$$a \to a_0, \quad b \to b_0 \quad \text{as} \quad \zeta \to -\infty \qquad (3.28)$$

and all derivatives of a and b vanish as $\zeta \to -\infty$. The scalar p in (3.22) represents an integrated pressure across a section of the pipe, and at $\zeta \to -\infty$ a pressure gradient is applied such that

$$-p' = 4\mu Q_0(a_0^2+b_0^2)/a_0^2b_0^2+a_0b_0U_0\omega\rho^*\sin(\omega t) \quad \text{as} \quad \zeta \to -\infty, \qquad (3.29)$$

where Q_0, u_0, ω are constants. Then, from (3.10), (3.17) to (3.19), (3.22) and (3.29) it follows that

$$\sigma\frac{\mathrm{d}Q}{\mathrm{d}\tau}+6Q = 6Q_0+\frac{3(a_0b_0)^{\frac{5}{2}}}{2(a_0^2+b_0^2)}\bar{R}\omega\sin\tau, \qquad (3.30)$$

where $$\bar{R} = \rho^*U_0(a_0b_0)^{\frac{1}{2}}/\mu, \quad \sigma = 2\rho^*\omega a_0^2b_0^2/(a_0^2+b_0^2), \quad \tau = \omega t. \qquad (3.31)$$

Equation (3.30) may be integrated to yield

$$Q = Q_0+\frac{3(a_0b_0)^{\frac{5}{2}}\bar{R}\omega}{2(a_0^2+b_0^2)(\sigma^2+36)}[6\sin(\omega t)-\sigma\cos(\omega t)]+K\exp(-6\omega t/\sigma). \qquad (3.32)$$

The constant K can be determined from initial conditions. Once the shape of the pipe is specified, the values of p at any section of the pipe may be found by a single integration of equation (3.22). Also, the values of the assigned forces $\boldsymbol{f}_c$ and $\boldsymbol{l}_{Nc}$ (these symbols were introduced in §2 following equation (2.9)) at the walls exerted on the fluid can be determined once the three-dimensional body force (which is taken to be zero here) is specified. It is interesting that in the special case when the pipe has circular sections with $a = b$, $a_0 = b_0$, because of the symmetry of the flow, the actual values of $\boldsymbol{f}_c$ and $\boldsymbol{l}_{Nc}$ can be readily determined. If the contact forces exerted by the wall on the fluid has tangential and normal components (F_t, F_n), then, in the absence of body forces,

$$\lambda \boldsymbol{f} = 2\pi a(F_t - a'F_n)\,\boldsymbol{e}_3, \quad \lambda \boldsymbol{l}_1 = \pi a^2(a'F_t + F_n)\,\boldsymbol{e}_1, \quad \lambda \boldsymbol{l}_2 = \pi a^2(a'F_t + F_n)\,\boldsymbol{e}_2. \quad (3.33)$$

The values of $\boldsymbol{f}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ may be found from (3.12) and (3.22) where A (or $2Q/\pi$) is given by (3.32). The results are then applicable to any symmetric flow in a circular pipe, subject to the conditions imposed on a at $\zeta = -\infty$. We note that one special case of the present solution is that discussed by Duck (1978) using a different procedure, when a has the value $a = a_0\{1 + \alpha c^2/(\zeta^2 + c^2)\}$, although the solution given by Duck has an approximation restricted to small values of α.

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Appendix A

Consider a three-dimensional body embedded in a euclidean 3-space and identify each material point of the body by a convected system of coordinates θ^i $(i = 1, 2, 3)$. Let $\boldsymbol{r}^*$ be the position vector, from a fixed origin, of a typical particle (material point) in the present configuration of the body at time t. Then,

$$\boldsymbol{r}^* = \boldsymbol{r}^*(\theta^i, t), \quad \boldsymbol{g}_i = \partial \boldsymbol{r}^*/\partial \theta^i, \quad \boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j,$$
$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j, \quad g^{ij} = \boldsymbol{g}^i \cdot \boldsymbol{g}^j, \quad g^{\frac{1}{2}} = [\boldsymbol{g}_1 \boldsymbol{g}_2 \boldsymbol{g}_3], \quad (\mathrm{A}\ 1)$$

where $\boldsymbol{g}_i, \boldsymbol{g}^i$ are covariant and contravariant base vectors, respectively, g_{ij} and g^{ij} are covariant and contravariant metric tensors, respectively, and δ^i_j is the Kronecker delta. The velocity vector of a typical particle of the body is denoted by $\boldsymbol{v}^*$ and

$$\boldsymbol{v}^* = \dot{\boldsymbol{r}}^*, \quad (\mathrm{A}\ 2)$$

where a superposed dot stands for the material time derivative holding θ^i fixed. For convenience we adopt the notation $\theta^3 = \theta$. We assume that the body is bounded by the surface

$$H(\theta^1, \theta^2, \theta) = 0 \quad (\mathrm{A}\ 3)$$

and suppose that this surface represents the lateral surface of a jet-like or rod-like body along the θ-direction.

Now, for the jet-like (or tube-like) body under consideration, let the position vector $\boldsymbol{r}^*$ admit the representation

$$\boldsymbol{r}^* = \sum_{N=0}^{K} \lambda_N(\theta^1, \theta^2)\,\boldsymbol{d}_N, \quad \lambda_0 = 1, \quad \boldsymbol{d}_0 = \boldsymbol{r}, \quad \boldsymbol{r} = \boldsymbol{r}(\theta, t), \quad \boldsymbol{d}_N = \boldsymbol{d}_N(\theta, t), \quad (\mathrm{A}\ 4)$$

where λ_N are functions of θ^1, θ^2. In (A 4), $\boldsymbol{r}$ can be identified as the position vector representing points on a one-dimensional curve in the jet-like (or tube-like) body, with its tangent vector defined by

$$\boldsymbol{a}_3 = \partial \boldsymbol{r}/\partial\theta, \quad a_{33} = \boldsymbol{a}_3 \cdot \boldsymbol{a}_3. \tag{A 5}$$

At points on the curve $(\mathrm{A}\ 4)_4$ the base vectors and metric tensors in (A 1) take the values

$$\boldsymbol{g}_i = \boldsymbol{a}_i(\theta, t), \quad \boldsymbol{g}^i = \boldsymbol{a}^i(\theta, t), \quad g_{ij} = a_{ij}, \quad g^{ij} = a^{ij}, \quad g = a. \tag{A 6}$$

With the help of (A 4), the velocity vector $\boldsymbol{v}^*$ defined by (A 2) takes the form

$$\boldsymbol{v}^* = \sum_{N=0}^{K} \lambda_N(\theta^1, \theta^2)\, \boldsymbol{w}_N, \quad \boldsymbol{v} = \boldsymbol{w}_0, \quad \boldsymbol{v} = \boldsymbol{v}(\theta, t) = \dot{\boldsymbol{r}}, \quad \boldsymbol{w}_N = \boldsymbol{w}_N(\theta, t) = \dot{\boldsymbol{d}}_N. \tag{A 7}$$

Let ζ^i $(i = 1, 2, 3)$ be a system of fixed curvilinear coordinates in the same euclidean 3-space and let points in this space be specified by a position vector $\bar{\boldsymbol{r}}^* = \bar{\boldsymbol{r}}^*(\zeta^i)$, with corresponding base vectors and metric tensors (all dependent on ζ^i only) given by

$$\bar{\boldsymbol{g}}_i = \partial \bar{\boldsymbol{r}}^*/\partial \zeta^i, \quad \bar{\boldsymbol{g}}^i \cdot \bar{\boldsymbol{g}}_j = \delta^i_j, \quad \bar{g}_{ij} = \bar{\boldsymbol{g}}_i \cdot \bar{\boldsymbol{g}}_j, \quad \bar{g}^{ij} = \bar{\boldsymbol{g}}^i \cdot \bar{\boldsymbol{g}}^j, \quad \bar{g}^{\frac{1}{2}} = [\bar{\boldsymbol{g}}_1 \bar{\boldsymbol{g}}_2 \bar{\boldsymbol{g}}_3]. \tag{A 8}$$

We select a fixed reference curve in this space which, with its tangent vector, is specified by

$$\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}(\zeta), \quad \bar{\boldsymbol{a}}_3 = \partial \bar{\boldsymbol{r}}/\partial \zeta, \quad \bar{a}_{33} = \bar{\boldsymbol{a}}_3 \cdot \bar{\boldsymbol{a}}_3, \quad \zeta^3 = \zeta. \tag{A 9}$$

At points on the fixed curve (A 9), the base vectors and metric tensors in (A 8) take the values

$$\bar{\boldsymbol{g}}_i = \bar{\boldsymbol{a}}_i(\zeta), \quad \bar{\boldsymbol{g}}^i = \bar{\boldsymbol{a}}^i(\zeta), \quad \bar{g}_{ij} = \bar{a}_{ij}, \quad \bar{g}^{ij} = \bar{a}^{ij}, \quad \bar{g} = \bar{a}. \tag{A 10}$$

In terms of the fixed coordinates ζ^i, the velocity of a point of the jet-like (tube-like) body at time t may be represented (in eulerian form) by

$$\boldsymbol{v}^* = \bar{\boldsymbol{v}}^*(\zeta^i, t) = \bar{v}^{*i} \bar{\boldsymbol{g}}_i = \sum_{N=0}^{K} \bar{\lambda}_N(\zeta^1, \zeta^2)\, \bar{\boldsymbol{w}}_N, \quad \bar{\lambda}_0 = 1, \quad \bar{\boldsymbol{v}} = \bar{\boldsymbol{v}}(\zeta, t) = \bar{\boldsymbol{w}}_0, \quad \bar{\boldsymbol{w}}_N = \bar{\boldsymbol{w}}_N(\zeta, t), \tag{A 11}$$

where $\bar{\lambda}_N$ are functions of ζ^1, ζ^2. The surface (A 3) which bounds the body is now specified by

$$\bar{H}(\zeta^1, \zeta^2, \zeta^3, t) = 0. \tag{A 12}$$

This is a material surface and moves with the body so that

$$\partial \bar{H}/\partial t + \bar{v}^{*i}\, \partial \bar{H}/\partial \zeta^i = 0. \tag{A 13}$$

Any function F^* associated with the body may be expressed either in terms of θ^i, t or ζ^i, t. Thus

$$F^*(\theta^i, t) = \bar{F}^*(\zeta^i, t). \tag{A 14}$$

In particular, the mass density has the representations

$$\rho^*(\theta^i, t) = \bar{\rho}^*(\zeta^i, t). \tag{A 15}$$

We note that

$$\left.\begin{aligned} \dot{\rho}^* + \rho^* \operatorname{div} \boldsymbol{v}^* &= \partial \bar{\rho}^*/\partial t + \operatorname{div}(\bar{\rho}^* \bar{\boldsymbol{v}}^*) \\ &= \partial \bar{\rho}^*/\partial t + \bar{g}^{-\frac{1}{2}} (\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{v}^{*i})_{,i}, \\ \overline{\rho^* F^*}^{\,\cdot} + \rho^* F^* \operatorname{div} \boldsymbol{v}^* &= \partial(\bar{\rho}^* \bar{F}^*)/\partial t + \operatorname{div}(\bar{\rho}^* \bar{F}^* \bar{\boldsymbol{v}}^*) \\ &= \partial(\bar{\rho}^* \bar{F}^*)/\partial t + \bar{g}^{-\frac{1}{2}} (\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{F}^* \bar{v}^{*i})_{,i} \end{aligned}\right\} \tag{A 16}$$

where $(\ )_{,i} = \partial(\ )/\partial \zeta^i$, and repeated lower case Latin indices are summed.

We now choose the convected coordinates θ^i such that at time t the θ^i-curves coincide with the fixed ζ^i-curves. Also, the moving curve represented by the vector $\boldsymbol{r}$ in (A 4) coincides with the fixed reference curve (A 9) at this time. With this choice of θ^i, we put:

$$\left.\begin{aligned} &\boldsymbol{g}_i = \bar{\boldsymbol{g}}_i, \quad g_{ij} = \bar{g}_{ij}, \quad g^{ij} = \bar{g}^{ij}, \quad g^{\frac{1}{2}} = \bar{g}^{\frac{1}{2}}, \\ &\boldsymbol{a}_i = \bar{\boldsymbol{a}}_i, \quad a_{ij} = \bar{a}_{ij}, \quad a^{ij} = \bar{a}^{ij}, \quad \lambda_N(\theta^1, \theta^2) = \bar{\lambda}_N(\zeta^1, \zeta^2). \end{aligned}\right\} \tag{A 17}$$

Multiplying both sides of (A 16) by $\lambda_N \lambda_M = \bar{\lambda}_N \bar{\lambda}_M$ we have

$$(\dot{\rho}^* + \rho^* \operatorname{div} \boldsymbol{v}^*)\, \lambda_N(\theta^1, \theta^2)\, \lambda_M(\theta^1, \theta^2) = \partial(\bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M)/\partial t$$
$$+ \bar{g}^{-\frac{1}{2}}(\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \bar{v}^{*i})_{,i} - \bar{\rho}^*[(\partial\bar{\lambda}_M/\partial\zeta^\alpha)\, \bar{\lambda}_N + (\partial\bar{\lambda}_N/\partial\zeta^\alpha)\, \bar{\lambda}_M]\, \bar{v}^{*\alpha}, \tag{A 18a}$$

or

$$\overline{\rho^* g^{\frac{1}{2}} \lambda_N \lambda_M}^{\,\cdot} = \frac{\partial}{\partial t}(\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M) + (\bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \bar{v}^{*i})_{,i} - \bar{g}^{\frac{1}{2}} \bar{\rho}^*[(\partial\bar{\lambda}_M/\partial\zeta^\alpha)\, \bar{\lambda}_N + (\partial\bar{\lambda}_N/\partial\zeta^\alpha)\, \bar{\lambda}_M]\, \bar{v}^{*\alpha}, \tag{A 18b}$$

where Greek indices take the values of 1, 2 and repeated indices are summed. Hence, integration of (A 18*b*) over an area $\theta =$ const. bounded by the curve $H(\theta^1, \theta^2, \theta) = 0$, $\theta =$ const., which at time t coincides with an area $\zeta =$ const. bounded by the curve $\bar{H}(\zeta^1, \zeta^2, \zeta, t) = 0$, $\zeta =$ const., and using the surface condition (A 13), yields

$$\overline{\lambda y_{MN}}^{\,\cdot} = \partial(\bar{\lambda} \bar{y}_{MN})/\partial t + \partial(\bar{\lambda} \bar{v}_{MN})/\partial\zeta - \bar{\lambda}(\bar{u}_{MN} + \bar{u}_{NM}) = 0, \tag{A 19}$$

where

$$\left.\begin{aligned} &\lambda = \rho a_{33}^{\frac{1}{2}}, \quad \bar{\lambda} = \bar{\rho} \bar{a}_{33}^{\frac{1}{2}}, \quad y_{00} = \bar{y}_{00} = 1, \\ &\lambda y_{MN} = \iint g^{\frac{1}{2}} \rho^* \lambda_N \lambda_M \,\mathrm{d}\theta^1 \,\mathrm{d}\theta^2, \quad \bar{\lambda} \bar{y}_{MN} = \iint \bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \,\mathrm{d}\zeta^1 \,\mathrm{d}\zeta^2, \\ &\bar{\lambda} \bar{v}_{MN} = \iint \bar{g}^{\frac{1}{2}} \bar{\rho}^* \bar{\lambda}_M \bar{\lambda}_N \bar{v}^{*3} \,\mathrm{d}\zeta^1 \,\mathrm{d}\zeta^2, \\ &\bar{\lambda} \bar{u}_{MN} = \iint \bar{g}^{\frac{1}{2}} \bar{\rho}^* \left(\frac{\partial\bar{\lambda}_M}{\partial\zeta^\alpha}\right) \bar{\lambda}_N \bar{v}^{*\alpha} \,\mathrm{d}\zeta^1 \,\mathrm{d}\zeta^2. \end{aligned}\right\} \tag{A 20}$$

Similarly, multiplying $(\text{A } 16)_2$ by λ_N, integrating and using the surface condition (A 13), we obtain

$$\overline{\lambda F_N}^{\,\cdot} = \overline{\lambda \sum_{M=0}^{K} y_{MN} f_M}^{\,\cdot} = \frac{\partial}{\partial t}\left[\bar{\lambda} \sum_{M=0}^{K} \bar{y}_{MN} \bar{f}_M\right] + \frac{\partial}{\partial\zeta}\left[\bar{\lambda} \sum_{M=0}^{K} \bar{f}_M \bar{v}_{MN}\right] - \bar{\lambda} \sum_{M=0}^{K} \bar{f}_M \bar{u}_{NM}. \tag{A 21}$$

The quantities f_M, $\bar{f}_M$ and F_N in (A 21) are related to F^* through

$$F^* = \sum_{M=0}^{K} f_M \lambda_M, \quad \bar{F}^* = \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M, \quad \lambda F_N = \iint g^{\frac{1}{2}} \rho^* F^* \lambda_N \,\mathrm{d}\theta^1 \,\mathrm{d}\theta^2 = \sum_{M=0}^{K} y_{MN} f_N, \tag{A 22}$$

where in writing $(\text{A } 22)_3$ we have also used $(\text{A } 20)_4$ and $(\text{A } 22)_2$.

Let $\alpha \leqslant \theta \leqslant \beta$ be an arbitrary part of the moving curve $\boldsymbol{r} = \boldsymbol{r}(\theta, t)$ which coincides with a part $\bar{\alpha} \leqslant \zeta \leqslant \bar{\beta}$ of the fixed curve $\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}(\zeta)$. Then, in view of (A 21),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\alpha}^{\beta} \rho F_N \,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\alpha}^{\beta} \rho \sum_{M=0}^{K} y_{MN} f_M \,\mathrm{d}s$$
$$= \frac{\partial}{\partial t}\int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho} \sum_{M=0}^{K} \bar{y}_{MN} \bar{f}_M \,\mathrm{d}\bar{s} + \left[\bar{\lambda} \sum_{M=0}^{K} \bar{v}_{MN} \bar{f}_M\right]_{\bar{\alpha}}^{\bar{\beta}} - \int_{\bar{\alpha}}^{\bar{\beta}} \bar{\rho} \sum_{M=0}^{K} \bar{u}_{NM} \bar{f}_M \,\mathrm{d}\bar{s}, \quad \text{(A 23)}$$

where $\mathrm{d}s = a_{33}^{\frac{1}{2}}\,\mathrm{d}\theta$, $\mathrm{d}\bar{s} = \bar{a}_{33}^{\frac{1}{2}}\,\mathrm{d}\zeta$. The results (A 21) and (A 23) are applied to the following particular cases in the main text:

(i) $$F^* = \boldsymbol{v}^*, \quad \bar{F}^* = \bar{\boldsymbol{v}}^*,$$

where $\boldsymbol{v}^*, \bar{\boldsymbol{v}}^*$ have the representations (A 7) and (A 11) and

(ii) $$F^* = \eta^* = \sum_{M=0}^{K} \eta_M \lambda_M, \quad \bar{F}^* = \bar{\eta}^* = \sum_{M=0}^{K} \bar{\eta}_M \bar{\lambda}_M,$$

where η^* is entropy density. In connection with entropy, one other form for weighted average of η^*, namely

$$\lambda \tilde{\eta}_N = \iint g^{\frac{1}{2}} \rho^* \eta^* \lambda_N \,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 = \lambda \sum_{M=0}^{K} y_{MN} \eta_M, \quad \text{(A 24)}$$

is needed for the developments in §2. Similarly, a further representation is needed for the internal energy, namely

$$\lambda \epsilon = \iint g^{\frac{1}{2}} \rho^* \epsilon^* \,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 = \sum_{M=0}^{K} \sum_{N=0}^{K} \lambda y_{MN} \epsilon_{MN},$$
$$\epsilon^* = \sum_{M=0}^{K} \sum_{N=0}^{K} \epsilon_{MN} \lambda_M \lambda_N, \quad \bar{\epsilon}^* = \sum_{M=0}^{K} \sum_{N=0}^{K} \bar{\epsilon}_{MN} \bar{\lambda}_M \bar{\lambda}_N, \quad \text{(A 25)}$$

so that using the forms (A 21) and (A 22) when $N = 0$ and $\lambda_0 = 1$ we have

$$\dot{\overline{\lambda \epsilon}} = \lambda \overline{\sum_{M=0}^{K} \sum_{N=0}^{K} y_{MN} \epsilon_{MN}}^{\,\boldsymbol{\cdot}}$$
$$= \frac{\partial}{\partial t}\left[\bar{\lambda} \sum_{M=0}^{K} \sum_{N=0}^{K} \bar{y}_{MN} \bar{\epsilon}_{MN}\right] + \frac{\partial}{\partial \zeta}\left[\bar{\lambda} \sum_{M=0}^{K} \sum_{N=0}^{K} \bar{v}_{MN} \bar{\epsilon}_{MN}\right]. \quad \text{(A 26)}$$

The foregoing analysis gives a connection between integral balances in lagrangian and eulerian forms, and hence direct connections between lagrangian and eulerian forms of field equations. These latter forms may be obtained by a more direct method. Multiplying each side of (A 14) by $\lambda_N(\theta^1, \theta^2) = \bar{\lambda}_N(\zeta^1, \zeta^2)$ and using (A 22) gives

$$\sum_{M=0}^{K} \rho^* \lambda_N \lambda_M \dot{f}_M = \sum_{M=0}^{K} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \frac{\partial \bar{f}_M}{\partial t} + \sum_{M=0}^{K} \bar{\rho}^* \bar{\lambda}_N \bar{\lambda}_M \bar{v}^{*3} \frac{\partial \bar{f}_M}{\partial \zeta} + \sum_{M=0}^{K} \bar{\rho}^* \left(\frac{\partial \bar{\lambda}_M}{\partial \zeta^\alpha}\right) \bar{\lambda}_N \bar{v}^{*\alpha} \bar{f}_M. \quad \text{(A 27)}$$

If we integrate over the surface $\theta = \text{const.}$, bounded by the curve $H(\theta^1, \theta^2, \theta, t) = 0$,

$\theta=$ const., or over the corresponding surface $\zeta=$ const. with which it coincides at time t, we obtain

$$\lambda \sum_{M=0}^{K} y_{MN} \dot{f}_M = \bar{\lambda} \sum_{M=0}^{K} \left[\bar{y}_{MN} \left(\frac{\partial \bar{f}_M}{\partial t} \right) + \bar{v}_{MN} \left(\frac{\partial \bar{f}_M}{\partial \zeta} \right) + \bar{u}_{MN} \bar{f}_M \right]. \qquad \text{(A 28)}$$

Equations of mass conservation, momentum, entropy and energy for the direct theory may be derived from the corresponding equations in three dimensions by methods similar to those used by Green & Naghdi (1979, 1985). Since the representations here are somewhat more general, the corresponding formulae for contact force vectors, contact director force vectors, etc., are recorded below. Let $\boldsymbol{T}$ be the stress tensor in the three-dimensional theory, $\boldsymbol{p}^*$ the entropy flux vector, both in the configuration of the body at time t, $\boldsymbol{f}^*$ the external body force, s^* the external rate of supply of entropy, ξ^* the internal rate of supply of entropy, and η^* the entropy density all per unit mass. Then, with

$$\boldsymbol{T}^i = \bar{g}^{\frac{1}{2}} \boldsymbol{T} \bar{\boldsymbol{g}}^i, \quad P^i = \bar{g}^{\frac{1}{2}} \boldsymbol{p}^* \cdot \bar{\boldsymbol{g}}^i, \quad \mathrm{d}A = \mathrm{d}\zeta^1 \, \mathrm{d}\zeta^2, \qquad \text{(A 29)}$$

the kinetical quantities which occur in the equations of motion of §2 can have the interpretations

$$\boldsymbol{n} = \iint \boldsymbol{T}^3 \, \mathrm{d}A, \quad \boldsymbol{m}_N = \iint \boldsymbol{T}^3 \bar{\lambda}_N \, \mathrm{d}A, \quad \boldsymbol{k}_N = \iint \boldsymbol{T}^\beta \left(\frac{\partial \bar{\lambda}_N}{\partial \zeta^\beta} \right) \mathrm{d}A, \qquad \text{(A 30)}$$

$$\left.\begin{aligned} \lambda \boldsymbol{f} &= \iint \bar{\rho}^* \bar{g}^{\frac{1}{2}} \boldsymbol{f}^* \, \mathrm{d}A + \oint [(\boldsymbol{T}^1 - \bar{v}^1 \boldsymbol{T}^3) \, \mathrm{d}\zeta^2 - (\boldsymbol{T}^2 - \bar{v}^2 \boldsymbol{T}^3) \, \mathrm{d}\zeta^1], \\ \lambda \boldsymbol{l}_N &= \iint \bar{\rho}^* \bar{g}^{\frac{1}{2}} \bar{\lambda}_N \boldsymbol{f}^* \mathrm{d}A + \oint \bar{\lambda}_N [(\boldsymbol{T}^1 - \bar{v}^1 \boldsymbol{T}^3) \, \mathrm{d}\zeta^2 - (\boldsymbol{T}^2 - \bar{v}^2 \boldsymbol{T}^3) \, \mathrm{d}\zeta^1], \end{aligned}\right\} \qquad \text{(A 31)}$$

where repeated Greek indices are summed over the values 1, 2 and where the line integrals are taken along the curve

$$\bar{H}(\zeta^1, \zeta^2, \zeta, t) = 0, \quad \zeta = \text{const.}, \qquad \text{(A 32)}$$

$$\bar{v}^\alpha = \bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{g}}^\alpha \quad \text{and} \quad \bar{\boldsymbol{v}} = \bar{v}^\alpha \bar{\boldsymbol{g}}_\alpha + \bar{\boldsymbol{g}}_3 \qquad \text{(A 33)}$$

is a vector tangential to the surface (A 12) so that

$$\bar{v}^\alpha \, \partial \bar{H} / \partial \zeta^\alpha + \partial \bar{H} / \partial \zeta^3 = 0. \qquad \text{(A 34)}$$

Similarly, the various thermal entities which occur in equations (2.10), (2.11), (2.17) and (2.18) can have the interpretations

$$\left.\begin{aligned} k &= \iint P^3 \, \mathrm{d}A, \quad k_N = \iint P^3 \bar{\lambda}_N \, \mathrm{d}A, \\ \lambda s_N &= \iint \bar{\rho}^* \bar{g}^{\frac{1}{2}} s^* \, \mathrm{d}A - \oint \bar{\lambda}_N [(P^1 - \bar{v}^1 P^3) \, \mathrm{d}\zeta^2 - (P^2 - \bar{v}^2 P^3) \, \mathrm{d}\zeta^1], \\ \lambda \tilde{\eta}_N &= \iint \bar{\rho}^* \bar{g}^{\frac{1}{2}} \eta^* \bar{\lambda}_N \, \mathrm{d}A, \\ \lambda \xi_N &= \iint \rho^* \bar{g}^{\frac{1}{2}} \xi^* \bar{\lambda}_N \, \mathrm{d}A + \iint P^\alpha \left(\frac{\partial \bar{\lambda}_N}{\partial \zeta^\alpha} \right) \mathrm{d}A. \end{aligned}\right\} \qquad \text{(A 35)}$$

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